ON THE REFRACTION OF A CYLINDRICAL PULSE IN AN INFINITE ELASTIC MEDIUM WITH A SINGLE THIN LAYER

BY

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1. Introduction

The study of propagation of a elastic wave in an elastic medium with layers has a fundamental value for many seismic subjects. The problems of the reflection and the refraction of plane elastic wave have been investigated by many seismologists.[1] But these problems for cylindrical and spherical elastic waves have not been investigated except the Petrashen's works and some other works of the refraction of spherical wave by a single layer.[2] The effects of the thin layer on the propagation of elastic wave are of much complexity. Therefore, the seismogram cannot be exactly interpreted from the viewpoint of the geometrical optics, particularly, in case that the distances of the source and the point of observation from the thin layer are much shorter than that of the point of observation from the source.

In the following discussion, the problem of refraction in the elastic medium with the thin layer due to the action of a buried line source is set up and solved for the case of an impulsive SH line source by the Cagniard's integral transformation method.[3] Following L. Cagniard, a suitable distortion of the contour not only results in considerable analytical simplification but also yields an exact closed algebraic expression for the displacement of refracted wave through the thin layer as a function of time.

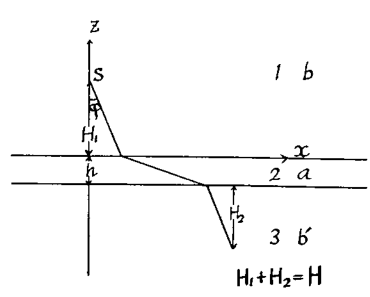

Fig. 1.

2. Formal solution

We shall consider the situation shown in Fig. 1. The single thin layer, 2 is sandwiched between the two semi-infinite elastic media, 1 and 3, and the line source is situated at (O, H_1). At time $t=0$, the line source disturbs the medium by emitting an axilly symmetric pulse.

We assume $a>b$ for the propagation velocities and the propagation velocities b and b' to be the same for simplicity. Then the displacements u_1 u_2 u_3 will be solutions of the following wave equations

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial z^2}-\frac{1}{b^2}\frac{\partial^2 u_1}{\partial t^2}=-\delta(x, z)f(t)\\
&\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial z^2}-\frac{1}{a^2}\frac{\partial^2 u_2}{\partial t^2}=0\\
&\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_3}{\partial z^2}-\frac{1}{b^2}\frac{\partial^2 u_3}{\partial t^2}=0\\
&f(t)=0 \qquad \text{for} \quad t>0 ,
\end{aligned} \tag{1}$$

where $-\delta(x, z)f(t)$ expresses the impulsive line SH source. The boundary conditions are

$$\begin{aligned}
&u_1=u_2 \qquad \mu_1\frac{\partial u_1}{\partial z}=\mu_2\frac{\partial u_2}{\partial z} \qquad \text{at} \quad z=0\\
&u_2=u_3 \qquad \mu_2\frac{\partial u_2}{\partial z}=\mu_3\frac{\partial u_3}{\partial z} \qquad \text{at} \quad z=-h .
\end{aligned} \tag{2}$$

Following L. Cagniard, all functions of time are subjected to a one sided Laplace transformation with respect to time. For the source function we have

$$F(s)=\int_0^\infty f(t)e^{-st}dt \tag{3}$$

and for the displacement

$$U(s)=\int_0^\infty u(x, z, t)^{-st}dt , \tag{4}$$

where s is a real, positive number large enough to ensure the convergence of integrals (3), (4).

Then, U_1, U_2, U_3 satisfy the differential equations

$$\frac{\partial^2 U_1}{\partial x^2}+\frac{\partial^2 U_1}{\partial z^2}-\frac{s^2}{b^2}U_1=-\delta(x, z)F(s) \tag{5_1}$$

$$\frac{\partial^2 U_2}{\partial x^2}+\frac{\partial^2 U_2}{\partial z^2}-\frac{s^2}{a^2}U_2=0 \tag{5_2}$$

$$\frac{\partial^2 U_3}{\partial x^2}+\frac{\partial^2 U_3}{\partial z^2}-\frac{s^2}{b^2}U_3=0 . \tag{5_3}$$

In the infinite medium U_0, the solution of (5_1), that is bounded as $|z|\to\infty$ is given by

$$\begin{aligned}
&U_0=\frac{F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(spx-s\beta|z|)}{2\beta}dp ,\\
&\alpha=\sqrt{a^{-2}-p^2} , \qquad \beta=\sqrt{b^{-2}-p^2} ,
\end{aligned} \tag{6}$$

where α, β are defined as that branches of the square roots at the right hand sides of (6) for which Re $(\alpha, \beta)\gtrless 0$. This implies that branch cuts are introduced along $\mathrm{Im}p=0$, $1/a<|\mathrm{Re}p|<\infty$, as Fig. 2.

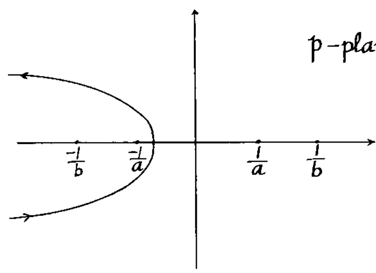

Fig. 2.

U_1, U_2, U_3 are expressed as following

$$\begin{aligned} U_1&=\frac{F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\beta}\left[e^{s(px-\beta|z-z_0|)}+Ge^{s(px-\beta|z+z_0|)}\right]dp \\ U_2&=\frac{F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\beta}\left[G_1e^{s(\alpha z-\beta z_0)}+G_2e^{-s(\alpha z+\beta z_0)}\right]dp \\ U_3&=\frac{F(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\beta}\left[G_3e^{s(px-\beta H-\alpha h)}\right]dp\,. \end{aligned} \tag{7}$$

Inserting U_1, U_2, U_3 into the boundary conditions (2) we obtain

$$\begin{aligned} G&=-\frac{(\mu_2^2\alpha^2-\mu_1^2\beta^2)}{\Delta}(1-e^{-2s\alpha h}) \\ G_1&=-\frac{2\mu_1\beta}{\Delta}(\mu_2\alpha+\mu_1\beta)e^{2s\alpha h} \\ G_2&=-\frac{2\mu_1\beta}{\Delta}(\mu_2\alpha-\mu_1\beta) \\ G_3&=1-\frac{(\mu_2\alpha-\mu_1\beta)^2}{\Delta}(1-e^{2s\alpha h}) \\ \Delta&=(\mu_2\alpha-\mu_1\beta)^2-(\mu_2\alpha+\mu_1\beta)^2e^{2s\alpha h} \end{aligned} \tag{8}$$

3. Transformation of the integrals and evaluation

In the following we investigate the refracted pulse u_3. Then we seperate the refracted field to individual pulses to be reflected 2n times at the boundary planes ($n=0, 1, 2, \cdots$). Namely, we note that $|(\mu_2\alpha-\mu_1\beta)/(\mu_2\alpha+\mu_1\beta)|<1$ and thus expand the formula for G_3

$$\frac{(\mu_2\alpha-\mu_1\beta)^2(1-e^{2s\alpha h})}{\Delta}=\left(\frac{\mu_2\alpha-\mu_1\beta}{\mu_2\alpha+\mu_1\beta}\right)^2-\frac{4\mu_1\mu_2\alpha\beta}{(\mu_2\alpha+\mu_1\beta)^2}\sum_{n=1}^{\infty}\left(\frac{\mu_2\alpha-\mu_1\beta}{\mu_2\alpha+\mu_1\beta}\right)^{2n}e^{-2ns\alpha h}. \tag{9}$$

Then,
$$\frac{G_3}{2\beta}=\frac{2\mu_1\mu_2\alpha}{(\mu_2\alpha+\mu_1\beta)^2}\left[1+\sum_{n=1}^{\infty}\left(\frac{\mu_2\alpha-\mu_1\beta}{\mu_2\alpha+\mu_1\beta}\right)^{2n}e^{-2ns\alpha h}\right] \tag{10}$$

Inserting the equation into (7) we apply the Cagniard's integral transformation method to each term corresponding to 2n times of reflection at the boundary planes. The Cagniard's procedure is to perform the integration on the p-plane along such a path that the right hand side of (7) can be recognized as the Laplace transform of a certain function of time. That is,

$$t=-px+\beta H+(1+2n)\alpha h \tag{11}$$

$$\frac{dp}{dt}=-\left\{x+\frac{pH}{\beta}+\frac{(1+2n)ph}{\alpha}\right\}^{-1} \tag{12}$$

where t is real and positive. By virtue of Cauchy's theorem and Jordan's lemma, the integral along the new path of (11) is equivalent to that along the original one. Taking into account the symmetry of the path of integation with respect to the real axis we obtain

$$(U_3)_n=Im\frac{F(s)}{2\pi}\int_{t_{0n}}^{\infty}\frac{(G_3)_n}{2\beta}e^{-st}\frac{dp}{dt}dt \tag{13}$$

$$t_{0n}=\frac{H}{b\cos\varphi}+\frac{(1+2n)\gamma h}{a\sqrt{\gamma^2-\sin^2\varphi}} \tag{14}$$

where t_{0n} is the arrival time of the first phase.

Applying the shift rule for Laplace transform to the function $F(s)\ exp(^{-st})$ in (13) and comparing this with (4) we obtain

$$(u_3)_n=\begin{cases}0 & (0<t<t_{0n})\\ \int_{t_{0n}}^{t}f(t-\tau)(\bar{u}_3)_n d\tau & (t_{0n}<t)\end{cases} \tag{15}$$

$$(\bar{u}_3)_n=Im\frac{1}{2\pi}\left\{\frac{4\mu_1\mu_2\alpha}{(\mu_2\alpha+\mu_1\beta)^2}\left(\frac{\mu_2\alpha-\mu_1\beta}{\mu_2\alpha+\mu_1\beta}\right)^{2n}\frac{dp}{dt}\right\}. \tag{16}$$

When $f(t)$ is a delta function,

$$(u_3)_n=(\bar{u}_3)_n. \tag{17}$$

4. Numerical results

We are interested in the time behaviour of the displacements of refracted pulses at different H and for different h of the thin layer so that the calculations have to be repeated many times with different values of the parameters. Thus, it was decided to perform the calculations with the electronic digital computer, KDC-1. The displacement was computed as a function of time for $f(t)=\delta(t)$, $a=2$, $b=1$ $\rho_1=\rho_3$, $x=10$, $h=0.5$, 1.0 and 2.0, and $H=2$, 3, $\cdots$, 20. The results are shown in Fig. 3, 4, 5.

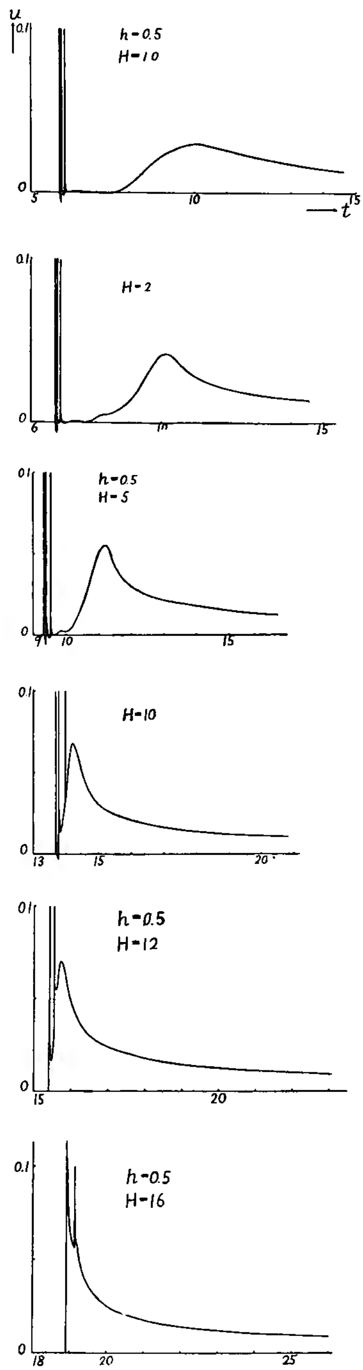

Fig. 3 Horizontal displacement u_3 at h=0.5.

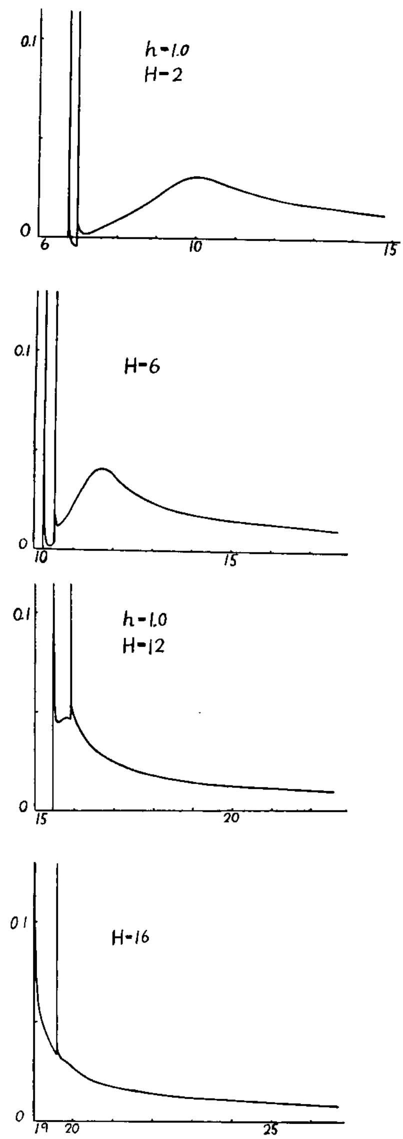

Fig. 4 Horizontal displacement u_3 at h=1.0.

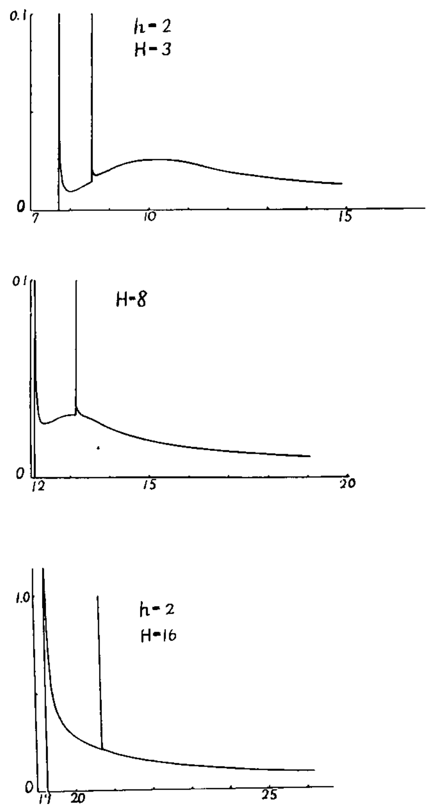

Fig. 5 Horizontal displacement u_3 at $h=2.0$

Except the directions corresponding to small values of φ, Fig. 3, 4, 5 show the appearances of two different groups. The phases of the first group appear at $t=t_{0n}$ $(n=0, 1, 2 \cdots)$, respectively and satisfy minimum time criterion and can be explained by the geometrical method. In case that the source function is $\delta(t)$, the displacements and the derivatives of the displacements with respect to time are infinite at $t=t_{0n}$. The phase of the second group which appears later than the first is not deducible by the geometrical method and is not well defined at arrival time and gradually increases and decreases. Thus the apparent period is much longer than the first. The phase of the second group gains in dominance with decreasing thickness h of the thin layer and vanishes when h tends to infinite. Therefore, this phase may be a kind of diffraction effect due to the interaction between the two boundary planes of the thin layer. Thus for the same value of h, the amplitude of the second phase increases with increasing H and at large value of H decreases a little and this phase is masked by the tail of the phase of the first group and thus is not well defined. The arrival time difference between two phases and apparent period of the second become small with increasing H.

As seen from the equation (15) for the arbitrary source function $f(t)$ and Fig. 3, 4, 5 for $\delta(t)$, the ratios of the amplitudes of the first refracted group to that of the second vary with the apparent period of incident pulse. Namely, when the apparent period of incident pulse is short, the amplitudes of the first pulses are larger than that of the second, and when the second pulse is short, inversly. Considering this character, we can observe seperately the two different phases by two seismographs with short band pass filters of different characters.

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